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Ultimate quantum limits for resolution of beam displacements

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Abstract. We compare two high sensitivity techniques which are used to measure very small displacements of physical objects by optical techniques: the interferometric devices, measuring longitudinal phase shifts, and the devices used to monitor transverse displacement of light beams. We detail the differences and the similarities for the quantum limits on the resolution of both systems. In both cases squeezed light can be used to resolve beyond the standard quantum limit and number correlated states allow us to reach the "Heisenberg" limit.

PACS. 42.50.-p Quantum optics – 42.30.-d Imaging and optical processing – 42.25.Hz Interference – 42.50.Ar Photon statistics and coherence theory – 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements – 42.50.Lc Quantum fluctuations, quantum noise, and quantum jumps

1 Introduction

Optics provides very sensitive methods to measure ultra-small relative displacements of physical objects. Interferometric techniques are in particular widely used in such measurements, and the very large Michelson interferometers currently built in different countries to detect gravitational waves [1] represent the ultimate achievements in this domain. Another highly sensitive technique, somehow easier to implement, is to measure the transverse relative displacement between a focussed light beam and a split detector by differential techniques. This technique, which reaches currently a nanometric sensitivity, is used in Atomic Force Microscopy to measure the small movements of the stylus, or to determine molecular motions in some biological applications [2]. In these measurements, as in any optical measurement, the ultimate sensitivity is limited by the quantum nature of the electromagnetic field. The case of the optical interferometer has been much studied, and it has been shown that the so-called shot noise or standard quantum limit is due to the vacuum fluctuations coupled to the interferometer and to the random motion of the mirrors induced by the radiation pressure fluctuations [3,4]. Shot noise sets also a limit in displacement measurement [5], which turns out to be, within a numerical factor of order one, the same as the shot noise limit for interferometers. It has been suggested [5] that this common limit arises from a fundamental similarity between the physics interferometry and beam displacement measurements.

It has been long appreciated that the shot noise limit, or standard quantum limit, does not represent the ultimate limit in optical measurements and that improvements are possible using non-classical states of light [6]. In particular, the improvement of interferometric measurements by nonclassical light has been indeed at the heart of the development of modern quantum optics [7–12]. It has been more recently realized that quantum effects play also an important role in the spatial properties of light [13,14]. This has opened a new chapter of quantum optics, usually labeled as *quantum imaging* [15]. Spatial quantum effects limit our ability to resolve optical images [16,17] and to measure transverse displacements of an optical beam [18,19]. It has recently been shown, both theoretically [18] and experimentally [19], that multimode non-classical states of light produced using a vacuum squeezed state can indeed be employed to resolve beam displacement below the standard quantum limit.

It is therefore of interest to ask in a more general way whether non-classical states of light of any kind can improve the sensitivity of beam displacement measurements in the same way as has been demonstrated for interferometry. This is the purpose of this paper, in which we compare the resolution that can be achieved in interferometry and beam displacement measurements by using different kinds of non-classical states of light. After recalling the results about the standard quantum limit in both cases (relative sensitivity proportional, as is well known, to $N^{-\frac{1}{2}}$, where N is the total number of photons measured in the experiment), we derive the ultimate limit that may be achieved using squeezed states of light (relative sensitivity

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proportional to $N^{-\frac{3}{4}}$), and finally we introduce a scheme capable of resolving beam displacements at the so-called "Heisenberg" limit [17], yielding a relative sensitivity proportional to N^{-1}.

2 Quantum noise in interferometry

There is a wide variety of practical interferometers. For the purposes of this paper, however, it suffices to consider only a very simplified quantum description in which two input modes, with annihilation operators $\hat{a}_{\rm in}$ and $\hat{b}_{\rm in}$, are superposed to give two output modes with annihilation operators $\hat{a}_{\rm out}$ and $\hat{b}_{\rm out}$. The relationship between the input and output operators will depend on the relative phase ϕ associated with the two arms of the interferometer. We suppose that the interferometer is adjusted such that this relationship is

$$\begin{aligned}\hat{a}_{\rm out} &= \cos\left(\frac{\phi}{2}\right)\hat{a}_{\rm in} + \sin\left(\frac{\phi}{2}\right)\hat{b}_{\rm in}\\ \hat{b}_{\rm out} &= \cos\left(\frac{\phi}{2}\right)\hat{b}_{\rm in} - \sin\left(\frac{\phi}{2}\right)\hat{a}_{\rm in}.\end{aligned} \tag{1}$$

This dependence on the relative phase allows us to detect changes in the phase by measuring the number of photons in each of the two output modes. For example, the output modes are simply the respective input modes if the relative phase shift is 0 and the modes are interchanged if $\phi = \pi$. A more complete introduction to the quantum theory of the interferometer can be found in [20].

2.1 Standard quantum limit

Let us assume for the sake of simplicity that the mirrors of the interferometer are infinitely massive. In this case, the fluctuations in radiation pressure [3,4,8] do not play any role. The shot noise or standard quantum limit in the phase shift measurement can then be understood by invoking either the partition noise in the distribution of photons between the output modes, or the interference with the vacuum field associated with the unused input interferometer mode [7]. From the perspective of our simple model, however, it is more helpful to use this second approach.

The standard quantum limit to resolution is reached when one input mode ($a_{\rm in}$) is prepared in a coherent state $|\alpha\rangle$ [20,21], with the remaining input mode ($b_{\rm in}$) left in its vacuum state. We can determine the limiting resolution in this case by finding the signal to noise ratio. The value of the phase shift for which this quantity is unity then provides a figure of merit for our phase resolution. The signal is obtained by measuring the difference in the numbers of photons detected in the two output modes. If the phase is shifted by $\Delta\phi$ from its original value ϕ then the expectation value of this difference in photon number is

$$\langle \hat{a}^\dagger_{\rm out}\hat{a}_{\rm out}\rangle - \langle \hat{b}^\dagger_{\rm out}\hat{b}_{\rm out}\rangle = |\alpha|^2\cos(\phi+\Delta\phi). \tag{2}$$

We can maximise sensitivity to the phase shift by choosing the operating point $\phi = -\pi/2$. For small phase shifts this gives the signal

$$\langle \hat{a}^\dagger_{\rm out}\hat{a}_{\rm out}\rangle - \langle \hat{b}^\dagger_{\rm out}\hat{b}_{\rm out}\rangle = N\Delta\phi, \tag{3}$$

where $N = |\alpha|^2$ is the mean number of photons. We could also choose $\phi = \pi/2$, in which case the signal (3) changes sign. The noise level is set by the uncertainty in the difference in photon numbers recorded in the two output modes at the operating point ($\phi = -\pi/2, \Delta\phi = 0$). A straightforward calculation gives

$$\left\langle \left(\hat{a}^\dagger_{\rm out}\hat{a}_{\rm out} - \hat{b}^\dagger_{\rm out}\hat{b}_{\rm out}\right)^2\right\rangle = N. \tag{4}$$

The square root of this variance is the required uncertainty and sets noise level so that the signal to noise ratio is

$$\frac{\rm Signal}{\rm Noise} = \frac{N\Delta\phi}{N^{1/2}}\cdot \tag{5}$$

Setting this ratio equal to unity gives us a measure of the smallest detectable phase shift

$$\Delta\phi = N^{-1/2}, \tag{6}$$

which is the standard quantum limit for interferometry. This proportionality between the resolution and the inverse square root of the mean number of photons is a general characteristic of the standard quantum limit and appears in numerous optical measurements.

2.2 Squeezed states

The standard quantum limit can be beaten if we replace the vacuum state of the unused input mode with a squeezed vacuum state [7,10]. The most straightforward way to see this is to repeat our calculation of the previous section, but without specifying the state of input mode $b_{\rm in}$. We retain the input mode $a_{\rm in}$ in the coherent state $|\alpha\rangle$ and the operating point $\phi = -\pi/2$. Under these conditions, the signal associated with small values of $\Delta\phi$ is

$$\begin{aligned}\langle \hat{a}^\dagger_{\rm out}\hat{a}_{\rm out}\rangle - \langle \hat{b}^\dagger_{\rm out}\hat{b}_{\rm out}\rangle &= \Delta\phi(|\alpha|^2 - \langle \hat{b}^\dagger_{\rm in}\hat{b}_{\rm in}\rangle)\\ &\quad - |\alpha|\langle \hat{b}_{\rm in}{\rm e}^{-{\rm i}\theta} + \hat{b}^\dagger_{\rm in}{\rm e}^{{\rm i}\theta}\rangle,\end{aligned} \tag{7}$$

where $\theta = \arg(\alpha)$. Hence the signal is linearly dependent on the phase shift $\Delta\phi$. Indeed, it will be proportional to $\Delta\phi$ for all states for which $\langle b_{\rm in}\rangle = 0$. These states include the vacuum and squeezed vacuum states [20,21]. We note that $|\alpha|^2$ is the mean number of photons in mode $a_{\rm in}$ and therefore the signal depends on the *difference* in the number of photons in the two input modes. The noise level is again set by the uncertainty in the difference in photon numbers in the two output modes at the operating point. If we specialise to states for which $\langle b_{\rm in}\rangle = 0$ then we find that the variance in the difference between the output mode photon numbers is

$$\langle(\hat{a}_{\rm out}^{\dagger}\hat{a}_{\rm out} - \hat{b}_{\rm out}^{\dagger}\hat{b}_{\rm out})^2\rangle = |\alpha|^2\langle(\hat{b}_{\rm in}{\rm e}^{-{\rm i}\theta} + \hat{b}_{\rm in}^{\dagger}{\rm e}^{{\rm i}\theta})^2\rangle + \langle\hat{b}_{\rm in}^{\dagger}\hat{b}_{\rm in}\rangle \cdot \quad (8)$$

We see here that the noise associated with the standard quantum limit arises from the vacuum fluctuations or uncertainty in the field operator $\hat{b}_{\rm in}{\rm e}^{-{\rm i}\theta}+\hat{b}_{\rm in}^{\dagger}{\rm e}^{{\rm i}\theta}$, which satisfy:

$$\langle 0|(\hat{b}_{\rm in}{\rm e}^{-{\rm i}\theta} + \hat{b}_{\rm in}^{\dagger}{\rm e}^{{\rm i}\theta})^2|0\rangle = 1. \quad (9)$$

We can reduce the noise level by preparing mode $b_{\rm in}$ in a squeezed vacuum state $|\zeta\rangle$ chosen so that

$$\langle\zeta|(\hat{b}_{\rm in}{\rm e}^{-{\rm i}\theta} + \hat{b}_{\rm in}^{\dagger}{\rm e}^{{\rm i}\theta})^2|\zeta\rangle = {\rm e}^{-2r}, \quad (10)$$

where r is positive. The noise level (8) can clearly be reduced by using this squeezed state. We should note, however, that a mode prepared in the squeezed vacuum state will contain photons. Indeed, the mean photon number for the squeezed vacuum state is

$$\langle\hat{b}_{\rm in}^{\dagger}\hat{b}_{\rm in}\rangle = \sinh^2 r. \quad (11)$$

The signal to noise ratio is again given by the signal (7) to the square root of the variance (8):

$$\frac{\rm Signal}{\rm Noise} = \frac{(|\alpha|^2 - \sinh^2 r)\Delta\phi}{(|\alpha|^2{\rm e}^{-2r} + \sinh^2 r)^{1/2}} \cdot \quad (12)$$

Setting this ratio equal to one again gives us a measure of the smallest detectable phase shift. For moderate levels of squeezing, the number of photons in the coherent input mode will greatly exceed the number in the squeezed vacuum mode ($|\alpha|^2 \gg \sinh^2 r$). This leads to a simplified form for the signal to noise ratio and hence of the smallest detectable phase shift

$$\Delta\phi = \frac{1}{N^{1/2}{\rm e}^r}, \quad (13)$$

where we have used the approximation $N \approx |\alpha|^2$. This minimum resolvable phase shift is clearly reduced below the standard quantum limit by the same factor (${\rm e}^r$) by which the uncertainty on the squeezed input quadrature is reduced below the level associated with the vacuum. The improved phase resolution (13) has been observed in experiments [10,11]. In seeking the ultimate limits, however, we should consider the possibility of very strongly squeezed light, with $r \gg 1$, and attempt to find the minimum resolvable phase. We can find this by returning to the general expression (12) and optimising the division of the mean photon number between the coherent and squeezed modes. For large numbers of photons, $N \gg 1$, we find that the optimum is given by $\sinh^2 r \approx {\rm e}^{2r}/4 \approx \sqrt{N}/2$. This gives the limit to the phase resolution attainable using squeezed light [7]

$$\Delta\phi = N^{-3/4}. \quad (14)$$

2.3 Equal intensity input modes

This squeezed-state limit is not the ultimate limit. It has been shown that a phase resolution that is inversely proportional to N should be possible [12]. This limit is often referred to as the "Heisenberg" limit because of the apparent uncertainty relation between photon number and phase. This qualitative idea has been given a rigorous meaning by Summy and Pegg who derived the phase optimized states of light using the Hermitian optical phase operator [22]. A number of proposals have been made for realising interferometry at the "Heisenberg" limit. Here, we concentrate on the one by Holland and Burnett [12] in which the two input modes are prepared with a precisely equal number of photons. It is sufficient for our purposes to consider only the case in which both input modes contain precisely $N/2$ photons, $|N/2\rangle_+|N/2\rangle_-$. At first sight this might seem strange as equation (7) suggests that we should keep the difference in the number of photons in the input modes as large as possible. In order to reach the optimal resolution, however, will have to work at a different operating point and also give up any knowledge of the sign of the phase shift. It is easier to demonstrate interferometry with these states in the Schrödinger picture rather than in the Heisenberg picture used previously. This means performing the unitary transformation inherent in (1) on the input state. This gives the output state of the two modes

$$|{\rm Out}\rangle = \exp\left(\frac{\phi + \Delta\phi}{2}\left(\hat{a}^{\dagger}\hat{b} - \hat{b}^{\dagger}\hat{a}\right)\right)|N/2\rangle_a|N/2\rangle_b. \quad (15)$$

We choose the operating point $\phi = 0$, so that, in the absence of any phase shift, each of the output modes contains precisely $N/2$ photons. We will be able to detect small phase shifts by any deviations from exact equality in the number of photons detected in the output modes. The probability that $2q$ more photons are detected in output mode a than in mode b, given the phase shift $\Delta\phi$ is simply [12]

$$\begin{aligned} P(2q|\Delta\phi) &= |_a\langle(N/2)+q|_b\langle(N/2)-q| \\ &\quad \times \exp(\frac{\Delta\phi}{2}(\hat{a}^{\dagger}\hat{b} - \hat{b}^{\dagger}\hat{a}))|N/2\rangle_a|N/2\rangle_b|^2 \\ &\approx J_q^2\left(\frac{N\Delta\phi}{2}\right), \end{aligned} \quad (16)$$

where the approximation is valid for $q \ll N$ and J_q denotes the Bessel function of order q. A simple derivation of this expression is given in Appendix A. The measurement described here is rather different to that considered in the preceding sections and so we require a new figure of merit with which to determine the minimum resolvable phase shift. We recall that for zero phase shift, the numbers of photons in each output is exactly $N/2$ and that any phase shift will result is a deviation from this balance. We associate the minimum resolvable phase shift with a reduction in the probability for equal numbers of photons from 1 to 1/2. This means that we require the smallest value of

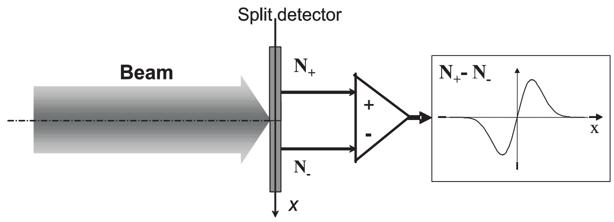

Fig. 1. Setup for beam displacement measurement.

$\Delta\phi$ for which $J_0^2(N\Delta\phi/2) = 1/2$. The resulting minimum resolvable phase shift is

$$\Delta\phi \approx \frac{2.24}{N}\,. \tag{17}$$

Naturally, measuring phase shifts in this way would be extremely sensitive to losses and finite detector efficiencies [23]. We should note, however, that an experiment has been performed with photon pairs ($N = 2$) [24].

3 Quantum noise in beam displacement measurements

The principle of a beam displacement measurement is sketched in Figure 1. A split detector is used, which measures the photon numbers N_+ and N_- of the two halves of an impinging beam. One monitors the intensity difference $N_+ - N_-$ between these two detectors, which are large compared to the size of beam to be used and also to the deflection to be detected. Assuming that the beam is initially centered on the split detector, the signal is zero, and any relative displacement in the x-direction between the detector and the beam gives rise to an imbalance between the two photodetectors, and therefore to a measurable signal on $N_+ - N_-$.

It has been pointed out that there is a natural connection between interferometry and such a displacement measurement [5]. In essence, this is because we can view beam deflection as a phase shift across the beam due to the fact that the deflection causes the path lengths to differ for rays on either side of the beam. We might expect, therefore, that the use of non-classical states of light would allow us to improve on the standard quantum limit for beam deflection measurements [18,19]. For the purposes of this paper, we require only a simple description of beam displacement and its measurement. The two detectors of the split detector can be considered to divide the x–y-plane into two parts. The *minus* detector counts photons in the half plane $x < 0$ and the *plus* detector counts photons in the half plane $x > 0$. The optical beam should be chosen to be propagating in the z-direction and to be symmetrical in x so that, for the centered beam, the same average number of photons will be detected in each side of the split detector. For definiteness, we consider a Gaussian beam with amplitude even in x:

$$u_{\rm e} = \frac{1}{\pi^{1/2} w_0} \exp\left(-\frac{x^2+y^2}{2w_0^2}\right)\cdot \tag{18}$$

It is helpful to also introduce a "flipped" or odd-Gaussian mode with an amplitude that changes sign at $x = 0$ [19]:

$$u_{\rm o} = \frac{1}{\pi^{1/2} w_0} \exp\left(-\frac{x^2+y^2}{2w_0^2}\right) \mathrm{sign}(x). \tag{19}$$

We note that this rather unlikely looking mode can be written as a well-behaved superposition of the more familiar odd-order Hermite-Gaussian modes:

$$u_{\rm o} = \exp\left(-\frac{x^2+y^2}{2w_0^2}\right) \sum_{n=0}^{\infty} H_{2n+1}\left(\frac{x}{w_0}\right) \frac{(-1)^n}{(2n+1)!n!2^{2n}\pi^{3/4}}\,. \tag{20}$$

The two sides of the split detector will be sensitive to the fields in the regions of positive or negative x (u_+ and u_- respectively). We can write these amplitudes as simple superpositions of $u_{\rm e}$ and $u_{\rm o}$:

$$u_+ = \frac{1}{\sqrt{2}}(u_{\rm e} + u_{\rm o})$$
$$u_- = \frac{1}{\sqrt{2}}(u_{\rm e} - u_{\rm o}). \tag{21}$$

We can assign annihilation operators to each of these modes and these will be related in the same way as the field amplitudes:

$$\hat{a}_+ = \frac{1}{\sqrt{2}}(\hat{a}_{\rm e} + \hat{a}_{\rm o})$$
$$\hat{a}_- = \frac{1}{\sqrt{2}}(\hat{a}_{\rm e} - \hat{a}_{\rm o}). \tag{22}$$

This is reminiscent of the relations (1) for an interferometer. The use of non-classical states of light to improve resolution relies on this similarity.

3.1 Standard quantum limit

The standard quantum limit is reached when the even mode $a_{\rm e}$ is prepared in a coherent state, with the odd mode $a_{\rm o}$ left in its vacuum state. As with interferometry, we can determine this limiting resolution by setting the signal to noise ratio equal to unity. The signal is obtained by measuring the difference in the numbers of photons detected in the two sides of the split detector. If the beam is displaced by the amount Δx, then this difference in photon numbers is

$$\langle \hat{a}_+^\dagger \hat{a}_+ \rangle - \langle \hat{a}_-^\dagger \hat{a}_- \rangle = |\alpha|^2 2 \int_0^{\Delta x} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y |u_{\rm e}|^2. \tag{23}$$

The integral corresponds to the proportion of the mode that is transfered between the two halves of the split detector by virtue of the beam displacement. For small displacements and the Gaussian mode (18) this gives the signal

$$\langle \hat{a}_+^\dagger \hat{a}_+ \rangle - \langle \hat{a}_-^\dagger \hat{a}_- \rangle = N \frac{2\Delta x}{\pi^{1/2} w_0}, \tag{24}$$

where $N = |\alpha|^2$. The noise level is set by the uncertainty in the difference in photon numbers recorded in the two halves of the split detector with the beam centred on $x = 0$. A straightforward calculation shows this to be

$$\left\langle (\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)^2 \right\rangle = N. \tag{25}$$

The square root of this variance is the required uncertainty and sets the noise level. Hence the signal to noise ratio for the beam displacement measurement is

$$\frac{\text{Signal}}{\text{Noise}} = \frac{2N\Delta x}{\pi^{1/2} w_0 N^{1/2}} \cdot \tag{26}$$

Setting this ratio equal to unity gives us a measure of the smallest detectable beam deflection

$$\Delta x = \left(\frac{\pi^{1/2} w_0}{2}\right) \frac{1}{N^{1/2}} \cdot \tag{27}$$

This is the standard quantum limit for a beam deflection measurement. As in interferometry, the attainable resolution is proportional to the inverse square root of the number of photons employed.

3.2 Squeezed states

The standard quantum limit can be beaten by preparing the odd mode a_0 in a squeezed vacuum state [18,19], with the coherent state retained for the even mode. This does not change the signal (24), but does modify the signal to noise ratio by reducing the noise. The variance in the difference between the numbers of photons detected in the two halves of the split detector for the undisplaced beam is

$$\langle (\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)^2 \rangle = |\alpha|^2 \langle (\hat{a}_0 \mathrm{e}^{-\mathrm{i}\theta} + \hat{a}_0^\dagger \mathrm{e}^{\mathrm{i}\theta})^2 \rangle + \langle \hat{a}_0^\dagger \hat{a}_0 \rangle, \tag{28}$$

where $\theta = \arg(\alpha)$. For a suitable choice of squeezed state, this variance becomes

$$\langle (\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)^2 \rangle = |\alpha|^2 \mathrm{e}^{-2r} + \sinh^2 r, \tag{29}$$

so that the signal to noise ratio is

$$\frac{\text{Signal}}{\text{Noise}} = \frac{2N\Delta x}{\pi^{1/2} w_0 (|\alpha|^2 \mathrm{e}^{-2r} + \sinh^2 r)^{1/2}} \cdot \tag{30}$$

Note that the numerator is proportional to the total mean number of photons $N = |\alpha|^2 + \sinh^2 r$, rather than to the difference between the number of coherent and squeezed photons found for the interferometer (12). This is because of the differing forms of the signal for interferometry and beam deflection, (7) and (24). The denominators, however, have the same form. For moderate levels of squeezing, for which $|\alpha|^2 \gg \sinh^2 r$, we can neglect $\sinh^2 r$ and also the difference between $|\alpha|^2$ and N. This leads to a simplified form for the smallest detectable beam displacement

$$\Delta x = \left(\frac{\pi^{1/2} w_0}{2}\right) \frac{1}{N^{1/2} \mathrm{e}^r}, \tag{31}$$

which represents an improvement over the standard quantum limit (27) by the factor e^r. This is the same factor by which moderate squeezing enabled us to surpass the standard quantum limit in interferometry. This improved resolution has recently been observed in experiment [19]. If very strongly squeezed light were available then we would have to take account of the number of photons in the squeezed mode and calculate the optimum level of squeezing. As with interferometry, we find that the optimum value $\sinh^2 r = \sqrt{N}/2$, which gives the limit to the beam displacement resolution attainable using squeezed light

$$\Delta x = \left(\frac{\pi^{1/2} w_0}{2}\right) \frac{1}{N^{3/4}} \cdot \tag{32}$$

Again we see the connection between interferometry and beam displacement measurements manifests itself in the quantum regime. In both cases, optimally squeezed light allows provides a resolution that betters the standard quantum limit by $N^{1/4}$.

3.3 Equal intensity input modes

We can use the results from interferometry to suggest how we might go beyond the squeezed state limit (32) for beam displacement measurements. For interferometers, we found that using modes with exactly equal numbers of photons allowed us to attain a phase resolution proportional the the inverse power of the total number of photons. We also had to give up knowledge of the sign of the phase shift. We will see that the same kind of state also allows us to reach a beam displacement resolution that is proportional to N^{-1}. For beam deflection measurements, however, we do not lose information about the direction of the displacement. We again work in the Schrödinger picture and prepare both the plus and minus modes (21) in a state containing precisely $N/2$ photons, $|N/2\rangle_+ |N/2\rangle_-$. This means that in the absence of any deflection, the two halves of the split detector will each register $N/2$ photocounts. We can then detect a beam displacement by a deviation from this exact equality in the number of photons. If the beam displacement Δx is positive, then the number of counts registered in the plus detector can only exceed the number registered in the minus detector. Each photon prepared in the mode a_+ will be detected in the plus detector, but each photon prepared in the minus detector will now be detected in the plus detector with a probability given by the overlap of the displaced minus mode with the plus detector:

$$p = \int_{-\infty}^{\infty} \mathrm{d}y \int_0^{\Delta x} \mathrm{d}x |u_\mathrm{e}|^2 \approx \frac{\Delta x}{\pi^{1/2} w_0} \cdot \tag{33}$$

The probability that $2q$ more photons are counted in the plus detector than in the minus detector, given that the beam displacement in Δx, is given by the Bernoulli sampling formula [21]

$$P(2q|\Delta x) = p^q (1-p)^{N-q} \frac{N!}{q!(N-q)!} \cdot \tag{34}$$

We can apply to this measurement the same figure of merit as we employed in our discussion of interferometry. We associate the minimum resolvable beam displacement with a reduction to 1/2 in the probability for equal numbers of photons being counted in the two halves of the split detector. This means that we require the value of Δx for which $[1 - \Delta x/(\pi^{1/2}w_0)]^N = 1/2$. We show in Appendix A that for large N this gives the minimum resolvable displacement

$$\Delta x = \left(\frac{\pi^{1/2}w_0}{2}\right)\frac{2\ln 2}{N}\,. \tag{35}$$

Once again we see the connection between interferometry and beam displacement measurements. In both cases, the best resolution is attained by using modes with precisely equal numbers of photons. An important difference is that the beam deflection measurement provides the sign of the displacement, but the interferometer does not give the sign of the phase shift. This is due to the different manner in which the signal is detected.

4 Conclusion

Both optical interferometry and beam displacement measurements have a standard quantum limit to resolution. This limit is proportional to the inverse square root of the number of photons employed. The standard quantum limit can be surpassed, however, by using squeezed states of light [7–11,18,19]. This relationship between interferometry and beam displacement measurements is not an accident, but rather arises from a simple physical connection between the two types of experiment [5]. We have demonstrated this connection for squeezed states of light, both for currently attainable moderate levels of squeezing and also for optimally squeezed light. We have also shown that the relationship persists as we seek the "Heisenberg" limit to sensitivity by using equal intensity states. It is probable that other schemes that have been proposed for interferometry beyond the standard quantum limit will also be applicable to improve beam deflection methods [25,26]. Squeezing and the use of quantum correlations have also been suggested as a means to improve measurement sensitivity on trapped atoms and ions [27–29] and Bose-Einstein condensates [30,31]. These proposals involve improving interferometry or frequency measurements. Our work suggests that such techniques might also prove useful in improving the resolution of imaging in coherent atom optics.

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Appendix A: Two approximations

We derive here two simple approximations used in the text to demonstrate the "Heisenberg" limits for interferometry [12] and for beam deflection. The first approximation is given in equation (16). We start by considering the operator in the exponent and noting that

$$\begin{aligned}\left(\hat{a}^\dagger\hat{b}\right)^n|N/2\rangle_a|N/2\rangle_b &= \sqrt{\frac{N}{2}+1}\cdots\sqrt{\frac{N}{2}+n}\sqrt{\frac{N}{2}}\cdots\\ &\times\sqrt{\frac{N}{2}+1-n}|(N/2)+n\rangle_a|(N/2)-n\rangle_b\\ &\approx (N/2)^n|(N/2)+n\rangle_a|(N/2)-n\rangle_b,\end{aligned} \tag{A.1}$$

where the approximation holds for $n \ll N$. Hence, with this restriction, we can write our operator in the approximate form

$$\frac{\Delta\phi}{2}(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a}) \approx \frac{N\Delta\phi}{4}(\hat{U} - \hat{U}^\dagger). \tag{A.2}$$

Here $\hat{U}$ is a unitary operator on the sub-space of states containing precisely N photons [32]

$$\begin{aligned}\hat{U} = \sum_{l=0}^{N-1}&|l+1\rangle_{aa}\langle l|\otimes|N-l-1\rangle_{bb}\\ &\times\langle N-l| + |0\rangle_{aa}\langle N|\otimes|N\rangle_{bb}\langle 0|.\end{aligned} \tag{A.3}$$

We can now approximate the unitary operator in (16) as

$$\begin{aligned}\exp\left(\frac{\Delta\phi}{2}(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})\right) &\approx \exp\left(\frac{N\Delta\phi}{4}(\hat{U} - \hat{U}^\dagger)\right)\\ &= \sum_{n=-\infty}^{\infty} J_n\left(\frac{N\Delta\phi}{2}\right)\hat{U}^n,\end{aligned} \tag{A.4}$$

where we have used the identity [33]

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} t^n J_n(x). \tag{A.5}$$

We should note that the infinite limits in (A.4) are only formally correct as our approximation holds only for $n \ll N$. The approximation in (16) now follows from the $n = q$ term in (A.4). It is also possible, of course, to derive this approximation by explicitly calculating the matrix elements in (16) and then considering the limit $q \ll N$. Our second approximation is needed in order to obtain (35). We require the solution of the equation

$$\left(1 - \frac{\Delta x}{\pi^{1/2}w_0}\right)^N = 1/2. \tag{A.6}$$

We can write the left hand side as an exponential by employing the inequalities [34]

$$\mathrm{e}^x > \left(1+\frac{x}{y}\right)^y > \mathrm{e}^{\frac{xy}{x+y}}, \qquad \text{(A.7)}$$

which hold for positive values of x and y. We can apply this formula by writing $x = N\Delta x/(\pi^{1/2}w_0)$ and $y = N$. For large N, we expect the resolution to better the standard quantum limit so that $x \ll y$. In this limit both sides of (A.7) tend to e^x and we can approximate (A.6) as

$$\left(1-\frac{\Delta x}{\pi^{1/2}w_0}\right)^N \approx \exp\left(-\frac{N\Delta x}{\pi^{1/2}w_0}\right). \qquad \text{(A.8)}$$

Setting this equal to 1/2 gives the required minimum resolvable beam displacement (35).

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